SOME CONTRIBUTIONS TO THE HEAT CONDUCTION AND THERMAL STRESSES ANALYSIS IN AIRCRAFT AND MISSILE STRUCTURES*

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Summary—The subject of the paper is the analysis of temperature distribution for bodies subjected to kinetic heating and heat losses by radiation, also in the case of boundary conditions depending upon time and upon surface temperature distributions.

In Part 1, the simple solution for the concentrated heat source in a heterogeneous body of whatever shape is found. By considering a surface layer of timewise and spacewise variable heat sources, the so-called "Q-Solution" is obtained. By combining the "Q-Solution" with the boundary conditions, an integrodifferential equation with respect to time is obtained, which can be easily integrated step by step, despite its non-linear nature.

Part II deals with the solution of typical problems. It contains the application of the theory to the hollow hemisphere, and to the hollow semicylinder. It also contains, for sake of comparison, the solution of a classical two-dimensional problem, of rather difficult analytical solution.

Part III contains numerical applications and results, including also evaluation of thermal stresses for the hollow hemisphere. Tables and graphs complete the work.

LIST OF SYMBOLS

REFERENCE DIMENSIONAL QUANTITIES

K^*	reference thermal conductivity
c^*	reference specific heat per unit volume
l	reference length
$\frac{l^2 c^*}{K^*}$	reference time
Q^*	reference heat flow per unit surface per unit time
$\frac{Q^*}{l}$	reference heat flow per unit volume per unit time
$\frac{K^*}{l}$	reference coefficient of kinetic heating

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$\frac{Q^{*}l}{K^{*}}$	reference temperature
$\frac{K^{*}}{l}\left(\frac{K^{*}}{lQ^{*}}\right)^{3}$	reference coefficient of radiation

NON-DIMENSIONAL QUANTITIES

K	non-dimensional thermal conductivity
c	non-dimensional specific heat per unit volume
t	non-dimensional time
x, y, z	non-dimensional rectangular co-ordinates
r, θ, ϕ	non-dimensional polar co-ordinates
r, θ, z	non-dimensional cylindrical co-ordinates
Q	non-dimensional heat flow per unit surface per unit time
q	non-dimensional heat flow per unit volume per unit time
h	non-dimensional coefficient of kinetic heating
σ	non-dimensional coefficient of radiation
T	non-dimensional temperature

SPECIAL SYMBOLS

Introduction

T_{aW}	recovery temperature
T_W	surface temperatures
$\mathscr{F}$	functional dependence of Q on T_W
σT_0^4	heat supplied to the body through radiation, per unit area per unit time

Part 1

V	body volume
W	body boundary
ν	normal to W
ϵ	part of V inside the body
Δ_K	linear differential operator
$\dot{T}_{\infty} = \frac{1}{\int_V c\,\mathrm{d}V}$	time derivative of uniform body temperature
T_E	$= T - t\dot{T}_{\infty}$
U_n	eigenfunctions of system (5.1)
p_n^2	eigenvalues of system (5.1)
a_n	coefficients of expansion of T_E
P, P'	generic points of the body
λ	dummy variable of integration with respect to time
$C(P, P'; t, \lambda)$	temperature produced at time t, at point P, by a step unit source generated at time λ at point P'
v	instantaneous point source of heat in P'

P_W, P'_W generic points of W
$T_W(P_W, t)$ temperature at point P_W, at time t
δ time interval
$m\delta$ generic time ($m = 0, 1, 2, \ldots$)
$[Q(P_W)]_m$ heat flow at P_W and $t = m\delta$
$[\Delta Q(P_W)]_i$ $[Q(P_W)]_i - [Q(P_W)]_{i-1}$
$f(P)$ non-uniform initial temperature (if any)
b_n coefficients of expansion of $f(P)$

Part 2

Hemisphere and Semicylinder

β non-dimensional inner radius of hemisphere or semi-cylinder (referred to outer radius)
A constant of normalization
$\left.\begin{matrix} C_s \\ D_s \end{matrix}\right\}$ functions of p_r
$\mathscr{P}_s$ sth Legendre polynomial
ϕ from (3.2)
Φ_s from (5.2)

Stiffener Section

a, η, η' dimensions of stiffener section
F_m, G_s indetermined constants
$\xi_m, \xi'_s, \rho_{ms}, \rho'_{sm}$ coefficients of compatibility equation

Appendices

P_s, Q_s functions of pr
ρ_m coefficient of ρ_{m0}
$\Delta_1, \Delta_2, \Omega', \Omega''$ functions of p for linearization

Part 3

g gravity acceleration
M Mach number
σ_r radial stress
σ_θ circumferential stress
$\left.\begin{matrix} Y_1(p) \\ Y_2(p) \end{matrix}\right\}$ linearized functions
t^* dimensional time (Figs. 3a.3, 3b.3)

INTRODUCTION

The design of an aircraft or missile is often ruled by the transient-temperatures of its external surface, due to kinetic heating and radiation.

At each point of the external surface W, the relationship

$$Q = h[T_{aW} - T_W] - \sigma(T_W{}^4 - T_0{}^4) = \mathscr{F}(t, T_W) \tag{1.I}$$

between temperature T_W and surface heat flow Q is customarily admitted(1, 2, 3). In equation (1.I), the symbol $\mathscr{F}$ denotes a functional dependence of the heat flow Q at any point of W on the temperatures at all points of W itself. In general, in fact, the parameters h, T_{aW}, σ, T_0 appearing in equation (1.I) must be considered to be four functions not only of time (as speed and altitude of the body are varying), but also of the unknown distribution of surface temperatures. When those four functions are given, e.g. from tests data, the problem arises how to study the heat conduction in the body with the appropriate boundary condition (1.I). Since of the complicated non-linear form of such boundary conditions, neither classical methods, nor those based on Laplace transform(4) seem to be adequate for the problem being considered(5).

The present analysis provides the solution for a heterogeneous body of whatever shape by means of an integro-differential equation which can be easily integrated step by step, the time being the only variable of integration.

Such integro-differential equation is obtained by combining equation (1.I) with a linear solution which will be hereafter called "Q-solution". The Q-solution provides the timewise variation of surface temperature due to whatever distribution of the surface heat flow Q. In the present analysis, the Q-solution is found to have a simple analytical expression which can be readily obtained, in explicit form, for whatever body. Such analytical expression is obtained by using a fundamental solution for unsteady heat conduction problems, which is also found in the paper.

The Q-solution can also be directly combined with wind-tunnel tests, the latter ones replacing equation (1.I), to get the solution to the problem. For the wind-tunnel tests of such analytical experimental procedure, only the similarity of the wind-tunnel flow field (including temperature and heat flows at the body surface), and not the thermal similarity inside the body, is required.

PART 1

A GENERAL APPROACH TO HEAT CONDUCTION PROBLEMS IN SOLIDS WITH VARIABLE AND NON-LINEAR BOUNDARY CONDITIONS

A Fundamental Solution for the Heterogeneous Finite Body

A body of volume V, bounded by one or more surfaces, denoted altogether as W, is considered. The body, initially at zero temperature, is unheated everywhere, except in a small volume ϵ, where, starting from $t = 0$, a constant heat flow equal to $1/\epsilon$ per unit volume per unit time, is applied.

At the time t, the variation from zero of the temperature T satisfies the equations:

$$q + \Delta_K T = c\,\frac{\partial T}{\partial t}; \qquad \begin{cases} q = 0 \text{ in } V - \epsilon \\ q = \dfrac{1}{\epsilon} \text{ in } \epsilon \end{cases} \tag{1.1}$$

$$\frac{\partial T}{\partial \nu} = 0 \text{ on } W; \qquad \text{for } t = 0;\ T = 0 \tag{2.1}$$

In equations (1.1)(2.1), c is the space-varying specific heat per unit volume; the operator Δ_K is expressed—e.g., in rectangular coordinates—by $(\partial/\partial x)[K(\partial/\partial x)] + (\partial/\partial y)[K(\partial/\partial y)] + (\partial/\partial z)[K(\partial/\partial z)]$, where K is the space-varying thermal conductivity of the body. Finally, ν denotes the normal to W.

The problem depicted by equations (1.1) and (2.1) is considered, first of all, as $t \to \infty$. After a very long time, a very great amount of heat has entered the body, and the temperature must be everywhere very great. It must also be almost uniformly distributed in the body because only finite, timely constant, temperature differences are required to let the finite heat flow q enter the body. Since the time derivative of the uniform temperature of the body, as $t \to \infty$, is clearly given by $\dot{T}_\infty = (1/\int_V c\,\mathrm{d}V)$, the above considerations suggest that the solution T of (1.1) and (2.1) can be put in the form:

$$T = \dot{T}_\infty\, t + T_E = \frac{t}{\int_V c\,\mathrm{d}V} + T_E \tag{3.1}$$

It is seen that T_E must start from zero, and approach, as $t \to \infty$, a steady solution. Therefore the following expression is chosen:

$$T_E = \sum_{1}^{\infty}{}_n\, a_n U_n (1 - \exp - p_n{}^{2t}) \tag{4.1}$$

where the a_n's and $p_n{}^2$'s are constants, and the U_n's are functions of space variables only. By introducing (3.1) and (4.1) into the system (1.1) and (2.1), the following conditions are deducted:

$$\Delta_K U_n + c p_n{}^2 U_n = 0 \qquad \frac{\partial U_n}{\partial \nu} = 0 \qquad \text{at } W \qquad n = 1, 2, \ldots \tag{5.1}$$

$$\Delta_K \sum_{1}^{\infty}{}_n\, a_n U_n = c\dot{T}_\infty - q = \begin{cases} \dfrac{c}{\int_V c\,\mathrm{d}V} & \text{in } V - \epsilon \\ \dfrac{c}{\int_V c\,\mathrm{d}V} - \dfrac{1}{\epsilon} & \text{in } \epsilon \end{cases} \tag{6.1}$$

It is seen that the p_n^2's and the U_n's are the eigenvalues and, respectively, the eigenfunctions of the problem (5.1). Since K and c cannot become negative, the p_n^2's are always real and positive, and the series $\sum_{1}^{\infty} {}_n a_n U_n$ is generally convergent. The U_n's are supposed to be normalized so that:

$$\int_V c U_n U_s \, dV = \begin{cases} 0 & \text{if} \quad n \neq s \\ 1 & \text{if} \quad n = s \end{cases} \tag{7.1}$$

After introduction of equation (5.1), equation (6.1) becomes:

$$\sum_{1}^{\infty} {}_n a_n p_n^2 U_n = \begin{cases} -\dfrac{1}{\int_V c \, dV} & \text{in} \quad V - \epsilon \\ -\dfrac{1}{\int_V c \, dV} + \dfrac{1}{c\epsilon} & \text{in} \quad \epsilon \end{cases} \tag{8.1}$$

When the series (8.1) is regular enough, the coefficients a_n can be obtained by multiplying both sides of equation (8.1) itself by cU_n and performing the integration all over the body.

Remembering (7.1) it is thus got:

$$a_n p_n^2 = -\frac{1}{\int_V c \, dV} \int_V c U_n \, dV + \frac{1}{\epsilon} \int_\epsilon U_n \, dV \qquad n = 1, 2, \ldots$$

The integral, $\int_V cU_n \, dV$, is zero because equations (5.1) are satisfied by $p_n = 0$ and $U_n = 1/\sqrt{(\int_V c \, dV)}$ const. (also because of the theorem of divergence).

When $\epsilon \to 0$ around a point P', $1/\epsilon \int_\epsilon U_n \, dV \to U_n(P')$.

Therefore:

$$a_n = \frac{1}{p_n^2} U_n(P') \tag{9.1}$$

The above results can be summarized and generalized in the formula:

$$C(P, P'; t, \lambda) = \frac{t - \lambda}{\int_V c \, dV} + \sum_{1}^{\infty} {}_n \frac{U_n(P) U_n(P')}{p_n^2} \left\{ -1 \exp\left[-p_n^2 (t - \lambda) \right] \right\} \tag{10.1}$$

$C(P, P'; t, \lambda)$ is the increase of temperature produced at time t and point P, by a step unit heat source generated at time λ and point P'.

It is of interest to point out the fact that the part of the right side of equation (10.1) which is independent of t {i.e. $\sum_{1}^{\infty} {}_n [U_n(P) U_n(P')/p_n^2]$} has an expression quite analogous to the well-known formula providing the static influence function of a structure as a bilinear expression of modes of vibration of the structure itself.

When the series of the left side of (10.1) can be derived termwise, the instantaneous point source of heat in a finite heterogeneous point source is obtained:

$$v = \frac{1}{\int_V c \, \mathrm{d}V} + \sum_{1}^{\infty} {}_n U_n(P) U_n(P') \exp[-p_n{}^2 (t - \lambda)] \qquad (10'.1)$$

In the particular case where the body is an infinite homogeneous medium, from equation (10′.1) and by using Fourier's integral, the well-known solution given, e.g. by Ref. 4, (2) is obtained.

Formula (10.1) enables us to write the expression of the temperature $T(P, t)$ at time t and point P due to whatever time and space varying distribution $q(P', t)$ of the heat flow per unit time and volume:

$$T(P,t) = \int_V \mathrm{d}V' \left\{ C(P,P';t,\lambda=0) q(P',t=0) + \int_0^t \mathrm{d}\lambda \left[C(P,P';t,\lambda) \left(\frac{\partial q}{\partial t}\right)_{t=\lambda} \right] \right\} * \qquad (11.1)$$

From the final equations (10.1) and (11.1) it is seen that the general problem is solved once the problem (5.1) is solved. Several methods are available[6, 7] to determine the eigenvalues and the eigenfunctions of the problem (5.1) which is essentially the problem of the natural modes and frequencies of vibration of a particular free body whose stiffness and mass per unit volume are K and c respectively. In the following, a method of linearization[8, 9] to solve such a problem will be used. Details will be given in Part 2, Appendix 2.

The Q-Solution

Formula (11.1) is also adequate to solve the case where heat is supplied to the body through the boundary W, as in the problem of kinetic heating and radiation, and not in the body volume. In such a case, the heat flow q must be supposed to be zero everywhere, except in a solid layer of small thickness $\mathrm{d}\nu$ at the surface W, where $q = Q/\mathrm{d}\nu$.

Formula (11.1) then becomes:

$$T(P, t) = \int_W \mathrm{d}W' \left\{ C(P, P'_W; t, \lambda = 0) Q(P'_W, t = 0) + \right.$$
$$\left. + \int_0^t \mathrm{d}\lambda \left[C(P, P'_W; t, \lambda) \left(\frac{\partial Q}{\partial t}\right)_{t=\lambda} \right] \right\} \qquad (12.1)$$

* To the right side of Eq. (11.1) the expression

$$\frac{\int_V f(P) c \, \mathrm{d}V}{\int_V c \, \mathrm{d}V} + \sum_{1}^{\infty} {}_n b_n U_n \exp - p_n{}^2 t$$

$$b_n = \int_V c \, \mathrm{d}V \left[f(P) - \frac{\int_V f(P) c \, \mathrm{d}V}{\int_V c \, \mathrm{d}V} \right] U_n$$

must be added, if at $t = 0$ the initial temperature is not zero, but $f(P)$.

The temperature at a generic point P_W of the boundary W is given by:

$$T(P_W, t) = \int\limits_W \mathrm{d}W' \Big\{ C(P_W, P'_W; t, \lambda = 0) Q(P'_W, t = 0) + \\ + \int\limits_0^t \mathrm{d}\lambda \Big[C(P_W, P'_W; t, \lambda) \left(\frac{\partial Q}{\partial t}\right)_{t=\lambda} \Big] \Big\} \qquad (13.1)$$

The above solution (12.1) and (13.1) will be hereafter called the "Q-Solution".

The Integrodifferential Equation for Kinetic Heating and Radiation

The general problem of kinetic heating and radiation is solved simply by combining equation (1.I), Introduction, with equation (13.1) i.e. by supposing, in equation (13.1), Q to be given by equation (1.I). The resulting integrodifferential equation can be easily solved step by step starting from the initial constant temperature of the body.

The practical procedure can be as follows: equal intervals of time, of amplitude δ, are considered; it is assumed that, during the interval from $t = m\delta$ to $t = (m+1)\delta$, $Q(P_W)$ has the time constant value $[Q(P_W)]_m$ given by equation (1.I) where for T_W the values of surface temperature for $t = m\delta$ are introduced:

$$[Q(P_W)]_m = \mathscr{F}\{t = m\delta, (T_W)_{t=m\delta}\} \qquad (14.1)$$

Now equation (13.1) is written in the form of a summation extended to the time intervals of amplitude δ:

$$T_W(P_W, m\delta) = \int\limits_W \mathrm{d}W \left\{ \sum_{i=0}^{m-1} C(P_W, P'_W; t = m\delta, \lambda = i\delta)[\Delta Q(P'_W)]_i \right\} \qquad (15.1)$$

where $[\Delta Q(P'_W)]_i = [Q(P'_W)]_i - [Q(P'_W)]_{i-1,}$ being the Q_i's given by (14.1) with $Q_1 = 0$.

Thus, the method consists in employing equation (14.1) for $m = 0$; then equation (15.1) for $m = 1$; then equation (14.1) for $m = 1$; then equation (15.1) for $m = 2$; and so on.

An Analytical Experimental Approach using the Q-Solution

The procedure described above is dependent upon equation (14.1). The same procedure can be used, replacing equation (14.1) with wind-tunnel tests in steady conditions. In such tests the surface heat flows $[Q(P_W)]_m$ must be measured, while the surface temperatures must be kept so as to satisfy the given distribution $(T_W)_{t=m\delta.}$ These steady wind-tunnel tests do not require any internal similarity in the model, but the problem of controlling the surface temperatures must be solved.

An analytical-experimental investigation in this direction has been undertaken at the supersonic wind tunnel of the University of Rome.

PART 2

SOLUTION OF TYPICAL PROBLEMS

General Remarks

As seen in Part 1, the general problem of kinetic heating and radiation reduces to a simple step-by-step integration once the Q-Solution is known.

On the other hand, the Q-Solution is completely known once the eigenvalues and eigenfunctions are known. Therefore, the following articles will be devoted to finding such unknowns for typical bodies, of a special interest in missile and aircraft design problems.

The Hollow Hemisphere

A hollow homogeneous hemisphere is now considered ($K = c = 1$) and eigenfunctions symmetric with respect to a polar axis are studied. The outer radius is taken as reference length, l, and the non-dimensional inner radius is denoted as β. By employing polar spherical co-ordinates, equations (5.1) become:

$$\left.\begin{aligned}&\frac{\partial}{\partial r}\left(r^2\,\frac{\partial U}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial U}{\partial\theta}\right)+r^2p^2U=0\\&\text{for } r=1 \text{ and for } r=\beta,\ \frac{\partial U}{\partial r}=0\\&\text{for } \theta=0 \text{ and for } \theta=\frac{\pi}{2};\ \frac{\partial U}{\partial\theta}=0\end{aligned}\right\}\qquad(1.2)$$

The eigenfunctions are found to be:

$$U=A\mathscr{P}_s(\cos\theta)\left\{\frac{D_s(pr)}{pr}\cos\left[p(1-r+\phi)\right]+\frac{C_s(pr)}{pr}\sin\left[p(1-r+\phi)\right]\right\}$$

$$s=0,2,4,\ldots\qquad(2.2)$$

where A is the constant of normalization (App. 3), $\mathscr{P}_s(\cos\theta)$ is Legendre polynomial of order s, and $C_s(pr)\ D_s(pr)$ are rational functions of the argument pr, related to Bessel functions of order $(s+\frac{1}{2})$ and $-(s+\frac{1}{2})$, whose expressions are given in App. I.

In equation (2.2), it is:

$$\phi=\frac{1}{p}\tan^{-1}\Phi_s(p)\qquad(3.2)$$

$$p(1-\beta)+\tan^{-1}\frac{\Phi_s(p)-\Phi_s(p\beta)}{1+\Phi_s(p)\Phi_s(p\beta)}=j\pi\qquad(j=1,2,\ldots)\qquad(4.2)$$

being:

$$\Phi_s=\frac{C_s/pr-[d/d(pr)](D_s/pr)}{D_s/pr+[d/d(pr)](C_s/pr)}$$

It is to be noticed that, as $p \to \infty$, one has $\Phi_s \to 0$, and

$$\tan^{-1} \frac{\Phi_s(p) - \Phi_s(p\beta)}{1 + \Phi_s(p)\Phi_s(p\beta)} \to 0. \tag{5.2}$$

Thus it is seen that the left side of the transcendental equation (4.2) —which gives the eigenvalues p^2's—is reduced by the method of linearization (App. 2) to be the sum of a linear term plus a term which is damped out as $p \to \infty$.

In the particular case of purely radial flow ($s = 0; \mathscr{P}_0 = 1;\ C_0 = 0;\ D_0 = 1$) it is obtained; $\Phi_0 = (1/pr)$; $\phi = (1/p)\tan^{-1}(1/p)$.

The Hollow Infinite Circular Semicylinder

Also in this case $K = c = 1$; the outer radius is taken as reference length, the inner non-dimensional radius is β. By using cylindrical coordinates, equations (5.1) become:

$$\left.\begin{aligned} &\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + p^2 U = 0 \\ &\text{for } r = 1,\ r = \beta;\ \frac{\partial U}{\partial r} = 0 \\ &\text{for } \theta = \frac{\pi}{2},\ \theta = \frac{\pi}{2};\ \frac{\partial U}{\partial \theta} = 0 \end{aligned}\right\} \tag{6.2}$$

Equations (6.2) are the same as those providing frequencies and modes of vibration of a semi-annular membrane.

The solutions of equations (6.2) can be written:

$$U = A\begin{pmatrix}\cos s\theta \\ \sin s\theta\end{pmatrix}\left[\frac{D_s}{pr}\cos p(1 - r + \phi) + \frac{C_s}{pr}\sin p(1 - r + \phi)\right] \begin{matrix} s = 0, 2, 4, \dots \\ s = 1, 3, 5, \dots \end{matrix} \tag{7.2}$$

where again A is the constant of normalization, and—in this case—D_s and C_s denote infinite series related to the Bessel functions of integer order s, whose expressions are given in App. 1.

Since the expression in square brackets of equation (7.2) is formally the same as in equation (2.2), equations (3.2), (4.2), (5.2) are still formally valid.

Two-dimensional Stiffener Section

Figure 1 represents the section of a stiffener for aircraft or missile structures. The problem of transient temperature distribution in it has been studied in Ref. (10) by an approximate procedure, obtained by combining Laplace transform with Galerkin method*. The problem can be solved exactly by the method of Part 1; here eigenfunctions and eigenvalues are determined.

* In Ref. 10 it is stated that no analytical approach for such a problem seemed to be possible.

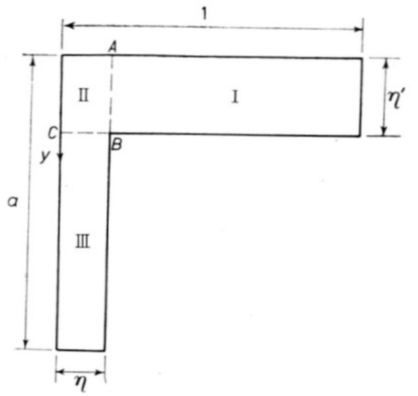


FIG. 1. Two-dimensional stiffener section.

Here again $K = c = 1$; then by referring to a system of rectangular co-ordinates x, y, and taking as reference length the length in x-direction of the upper leg (other symbols are clearly denoted in Fig. 1), equations (5.1) become:

$$\left.\begin{aligned} &\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + p^2 U = 0 \\ \text{all along the boundary} \quad &\frac{\partial U}{\partial \nu} = 0 \end{aligned}\right\} \tag{8.2}$$

Since of the particular shape of the body, each eigenfunction U is given by three separate expressions for the regions I, II, III (Fig. 1).

Equations (8.2) are the same as those providing modes and frequencies of vibration of a homogeneous free membrane having the same shape of the stiffener section.

In the regions I, II, III, of Fig. 1, the expressions of U are given by:

$$\left.\begin{aligned} (U)_{\mathrm{I}} &= A \sum_{0}^{\infty}{}_{m} F_m \cos\frac{m\pi y}{\eta'} \cos\left\{(1-x)\sqrt{\left[p^2 - \left(\frac{m\pi}{\eta'}\right)^2\right]}\right\} \\ (U)_{\mathrm{II}} &= A\left(\sum_{0}^{\infty}{}_{m} F_m \cos\frac{m\pi y}{\eta'} \cos\left\{x\sqrt{\left[p^2 - \left(\frac{m\pi}{\eta'}\right)^2\right]}\right\} \times \right. \\ &\qquad \frac{\sin\{(1-\eta)\sqrt{[p^2 - (m\pi/\eta')^2]}\}}{\sin\{\eta\sqrt{[p^2 - (m\pi/\eta')^2]}\}} + \\ &\quad + \sum_{0}^{\infty}{}_{s} G_s \cos\frac{s\pi x}{\eta} \cos\left\{y\sqrt{\left[p^2 - \left(\frac{s\pi}{\eta}\right)^2\right]}\right\} \times \\ &\qquad \left.\frac{\sin\{(a-\eta')\sqrt{[p^2 - (s\pi/\eta)^2]}\}}{\sin\{\eta'\sqrt{[p^2 - (s\pi/\eta)^2]}\}}\right) \\ (U)_{\mathrm{III}} &= A \sum_{0}^{\infty}{}_{s} G_s \cos\frac{s\pi x}{\eta} \cos\left\{(a-y)\sqrt{\left[p^2 - \left(\frac{s\pi}{\eta}\right)^2\right]}\right\} \end{aligned}\right\} \tag{9.2}$$

where G_s, F_m are indetermined constants and A is the constant of normalization (App. III).

It is easily proved that equations (9.2) comply with all the conditions of the problem, except the condition of continuity of the function U along the borderlines AB, BC (Fig. 1).

If such conditions are imposed, the linear system:

$$\left.\begin{aligned} &\xi_m(p)F_m + \sum_0^\infty{}_s\, \rho_{ms}(p)G_s = 0 \qquad && m = 0, 1, \ldots \\ &\xi'_s(p)G_s + \sum_0^\infty{}_m\, \rho'_{sm}(p)F_m = 0 \qquad && s = 0, 1, \ldots \end{aligned}\right\} \tag{10.2}$$

is obtained, where:

$$\left.\begin{aligned} \xi_m(p) &= \frac{\sin\{\sqrt{[p^2 - (m\pi/\eta')^2]}\}}{\sin\{\eta\sqrt{[p^2 - (m\pi/\eta')^2]}\}} \\ \xi'_s(p) &= \frac{\sin\{a\sqrt{[p^2 - (s\pi/\eta)^2]}\}}{\sin\{\eta'\sqrt{[p^2 - (s\pi/\eta)^2]}\}} \\ \rho_{ms} &= \frac{\epsilon_m(-1)^{m+s+1}\sin\{(a-\eta')\sqrt{[p^2 - (s\pi/\eta)^2]}\} \times \sqrt{[p^2 - (s\pi/\eta)^2]}}{\eta'\,[(m\pi/\eta')^2 + (s\pi/\eta)^2 - p^2]} \\ \rho'_{sm} &= \frac{\epsilon_s(-1)^{m+s+1}\sin\{(1-\eta)\sqrt{[p^2 - (m\pi/\eta')^2]}\} \times \sqrt{[p^2 - (m\pi/\eta')^2]}}{\eta[(m\pi/\eta')^2 + (s\pi/\eta)^2 - p^2]} \end{aligned}\right\} \tag{11.2}$$

$$\epsilon_m = \begin{cases} 1 & m = 0 \\ 2 & m \neq 0 \end{cases}$$

The eigenvalues p^2's are found from the determinantal equation of the system (10.2). A practical method for its solution is given in Appendix II; as customary, the method of linearization is employed (App. II).

Appendix 1

Values of the Functions $D_s(pr)$, $C_s(pr)$

Hollow Hemisphere

The functions $C_s(pr)$, $D_s(pr)$ are readily found through the recursion formulas:

$$\left.\begin{aligned} C_s &= -(pr)^s\left\{\frac{d}{d(pr)}\left[\frac{C_{s-1}}{(pr)^s}\right] + \frac{D_{s-1}}{(pr)^s}\right\} \\ D_s &= -(pr)^s\left\{\frac{d}{d(pr)}\left[\frac{D_{s-1}}{(pr)^s}\right] - \frac{C_{s-1}}{(pr)^s}\right\} \qquad s = 1, 2, \ldots \\ C_0 &= 0; \qquad D_0 = 1 \end{aligned}\right\} \tag{1.A}$$

Hollow Semicylinder

It is found:

$$\left.\begin{aligned} D_s(pr) &= pr[P_s(pr) + (-1)^s Q_s(pr)] \\ C_s(pr) &= pr[Q_s(pr) - (-1)^s P_s(pr)] \end{aligned}\right\} \tag{2.A}$$

where $P_s(pr)$ and $Q_s(pr)$ are, in the notations of Ref. 11, the coefficients of the expressions of Bessel functions as:

$$\left.\begin{aligned} J_s(pr)\sqrt{\left(\frac{1}{2}\pi pr\right)} &= P_s(pr)\cos\left[pr - \left(s+\frac{1}{2}\right)\frac{\pi}{2}\right] - \\ &\quad - Q_s(pr)\sin\left[pr - \left(s+\frac{1}{2}\right)\frac{\pi}{2}\right] \\ Y_s(pr)\sqrt{\left(\frac{1}{2}\pi pr\right)} &= P_s(pr)\sin\left[pr - \left(s+\frac{1}{2}\right)\frac{\pi}{2}\right] + \\ &\quad + Q_s(pr)\cos\left[pr - \left(s+\frac{1}{2}\right)\frac{\pi}{2}\right] \end{aligned}\right\} \tag{3.A}$$

From Ref. 11

$$P_s(pr) = 1 - \frac{(4s^2-1)(4s^2-9)}{2!(8pr)^2} + \frac{(4s^2-1)(4s^2-9)(4s^2-25)(4s^2-49)}{4!(8pr)^4} + \dots$$

$$Q_s(pr) = \frac{4s^2-1}{8pr} - \frac{(4s^2-1)(4s^2-9)(4s^2-25)}{3!(8pr)^3} + \dots$$

Appendix 2

Notes on the Method of "Linearization" for the Solution of Transcendental Equations

General Remarks

In the foregoing problems, the determination of eigenvalues always reduces to the solution of transcendental equations.

In order to avoid numerical difficulties, the author has developed a method of "linearization" that can be applied to many eigenvalue problems of mathematical physics(8, 9). This method simply consists in transforming the left side of the transcendental equation so as to reduce it to a linear term plus a quantity which is rapidly damped as the eigenvalue is increasing (hence the name of "linearization").

This allows a very simple procedure of interpolation, and the roots can be found by employing elementary means in a few minutes.

The method is now applied to the transcendental equations obtained in the foregoing articles.

Hollow Hemisphere and Hollow Semicylinder

When boundary conditions at $r = 1$ and $r = \beta$ are imposed, the following transcendental equation is obtained:

$$\sin[p(1-\beta)] + \frac{\Phi_s(p) - \Phi_s(p\beta)}{1 + \Phi_s(p)\Phi_s(p\beta)} \cos[p(1-\beta)] = 0 \qquad (4.A)$$

which can be written in the linearized form:

$$p(1-\beta) + \tan^{-1} \frac{\Phi_s(p) - \Phi_s(p\beta)}{1 + \Phi_s(p)\Phi_s(p\beta)} = j\pi; \qquad (j = 1, 2, \ldots) \quad (5.A)$$

The function which is under $\tan^{-1}$ is rapidly decreasing, as p increases, for all the cases considered. Thus the left side of equation (5.A) rapidly approaches the linear function $p(1-\beta)$.

(a) for the hollow hemisphere, it is found in general:

$$\lim_{p\to\infty} \Phi_s(p) = 0 \qquad (6.A)$$

In fact $C_s(p)$ is of the order of $1/p$, D_s is of the order of unity; so from equation (5.2) it is seen that $\Phi_s(p)$ is of the order of $1/p$. For instance,

$$\Phi_0(p) = \frac{1}{p}; \qquad \Phi_2(p) = \frac{1}{p}\frac{4 - 9/p^2}{1 - 9/p^2} \qquad (6.A)$$

(b) For the hollow semicylinder, it is found, again with the notations of Ref. 11:

$$\Phi_s(p) = \left[\frac{\left(Q_s - \frac{dP_s}{d(pr)} + \frac{P_s}{2pr}\right) - (-1)^s\left(P_s + \frac{dQ_s}{d(pr)} - \frac{Q_s}{2pr}\right)}{\left(P_s + \frac{dQ_s}{d(pr)} + \frac{Q_s}{2pr} + (-1)^s\left(Q_s - \frac{dP_s}{d(pr)} - \frac{P_s}{2pr}\right)\right.}\right]_{r=1} \qquad (7.A)$$

Now, since

$$\lim_{pr\to\infty} Q_s = \lim_{pr\to\infty} \frac{dQ_s}{d(pr)} = \lim_{pr\to\infty} \frac{dP_s}{d(pr)} = 0; \qquad \lim_{pr\to\infty} P_s = 1 \qquad (8.A)$$

it is found $\lim \Phi_s(p) = (-1)^{s+1}$, and the function under $\tan^{-1}$ in (5.A) tends to zero.

Two-dimensional Stiffener Section

For the sake of simplicity, a symmetric section ($a = 1$, $\eta = \eta'$) is considered, where symmetric and anti-symmetric eigenfunctions can be separately studied. For symmetric eigenfunctions $F_m = G_m$, and the two equations (10.2) reduce to one only.

Again for the sake of simplicity, the roots in the range $0 \leqslant p \leqslant \pi/\lambda$ are considered. Thus, from equation (11.2):

$$\left.\begin{aligned}
&\xi_0(p) = \frac{\sin p}{\sin(\eta p)}\,; \qquad \rho_{m0} = \bar{\rho}_m \sin p(1-\eta) \\
&\text{where } \bar{\rho}_m = \frac{\epsilon_m(-1)^{m+1}p}{\eta[(m\pi/\eta)^2 - p^2]} \\
&\xi_m(p) = \frac{\sinh\{\eta\sqrt{[(m\pi/\eta)^2 - p^2]}\}}{\sinh\{\sqrt{[(m\pi/\eta)^2 - p^2]}\}} \\
&\rho_{ms} = \frac{\epsilon_m(-1)^{m+s}\sinh\{(1-\eta)\sqrt{[(s\pi/\eta)^2 - p^2]}\}}{\eta[(m\pi/\eta)^2 + (s\pi/\eta)^2 - p^2]} \times \\
&\qquad\qquad \sqrt{\left[\left(\frac{s\pi}{\eta}\right)^2 - p^2\right]}
\end{aligned}\right\} \quad (9.A)$$

Thus the determinantal equation can be written:

$$\Delta_1 \frac{\sin p}{\sin \eta p} + \Delta_2 \sin p(1-\eta) = 0 \qquad (10.A)$$

where:

$$\left.\begin{aligned}
\Delta_1 = \Delta_1(p) = &\begin{vmatrix} \xi_1 + \rho_{11}; & \rho_{12}; & \cdots \\ \rho_{21}; & \xi_2 + \rho_{22}; & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} \\
\Delta_2 = \Delta_2(p) = &\begin{vmatrix} \bar{\rho}_0; & \rho_{01}; & \rho_{02}; & \cdots \\ \bar{\rho}_1; & \xi_1 + \rho_{11}; & \rho_{12}; & \cdots \\ \bar{\rho}_2; & \rho_{21}; & \xi_2 + \rho_{22}; & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}
\end{aligned}\right. \quad (11.A)$$

are functions of p which have no points of infinity, or discontinuity or oscillations. Now, letting

$$\Omega'(p) = \tan^{-1}\frac{\Delta_2}{2\Delta_1}\,; \qquad \Omega''(p) = \sin^{-1}[\sin \Omega' \cos p(1-2\eta)] \qquad (12.A)$$

Equation (10.A) reduces to the two equations:

$$\left.\begin{aligned}
p - \Omega' + \Omega'' &= 2j\pi & j &= 1, 2, \ldots \\
p - \Omega' - \Omega'' &= (2j+1)\pi & j &= 0, 1, \ldots
\end{aligned}\right\} \quad (13.A)$$

The functions Ω', Ω'' are rapidly damped; therefore equations (13.A) are the linearized form of the proposed transcendental equation.

Analogously anti-symmetric eigenvalues can be treated.

For non-symmetric sections, or values of $p \geqslant \pi/\lambda$, the method can still be applied, although, obviously, in a more complicated way.

APPENDIX 3

Constants of Normalization

(a) Hollow hemisphere

$$\frac{1}{A^2} = \frac{2\pi}{2s+1}\frac{1}{2p^2}\left\{[D(pr)\cos p(1-r+\phi) + C(pr)\sin p(1-r+\phi)]^2 \times \left[1 - \frac{s(s+1)}{(pr)^2}\right]\right\}; \qquad r=1; \qquad r=\beta \qquad (14.A)$$

(b) Hollow semicylinder

$$\frac{1}{A^2} = \frac{\pi}{2}\left\{[D(pr)\cos p(1-r+\phi) + C(pr)\sin p(1-r+\phi)] \times \left[1 - \frac{s^2}{(pr)^2}\right]\right\}; \qquad r=1; \qquad r=\beta \qquad (15.A)$$

(c) Stiffener section:

$$\frac{1}{A^2} = \frac{\eta'}{4}\sum_{0}^{\infty}{}_m F_m{}^2\left\{1 + \frac{\sin 2\sqrt{[p^2-(m\pi/\eta')^2]}}{2\sqrt{[p^2-(m\pi/\eta')^2]}}\right\} +$$

$$+ \frac{\eta a}{4}\sum_{0}^{\infty}{}_s G_s{}^2\left\{1 + \frac{\sin 2a\sqrt{[p^2-(s\pi/\eta)^2]}}{2a\sqrt{[p^2-(s\pi/\eta)^2]}}\right\} +$$

$$+ 2\sum_{0}^{\infty}{}_m \sum_{0}^{\infty}{}_s (-1)^{m+s}F_m G_s \frac{[p^2-(s\pi/\eta)^2][p^2-(m\pi/\eta')^2]}{[(m\pi/\eta')^2+(s\pi/\eta)^2-p^2]^2} \times$$

$$\frac{\sin\{a\sqrt{[p^2-(s\pi/\eta)^2]}\}\sin\{(a-\eta')\sqrt{[p^2-(s\pi/\eta)^2]}\}}{\sin\{\eta'\sqrt{[p^2-(s\pi/\eta)^2]}\}} \times$$

$$\frac{\sin\{\sqrt{[p^2-(m\pi/\eta')^2]}\}\sin\{(1-\eta)\sqrt{[p^2-(m\pi/\eta')^2]}\}}{\sin\{\eta\sqrt{[p^2-(m\pi/\eta')^2]}\}} \qquad (16.A)$$

PART 3

RESULTS AND APPLICATIONS

(a) *Transient Temperature and Thermal Stresses in a Missile Head*

Geometrical feature—Flight path. The general theory of Part 2, Art. 2, has been applied to the analysis of the transient temperature and thermal stresses distribution in a missile head of hemispherical shape. The material is assumed to be steel, and the external radius is 0·5 m = 19·7 in. Two cases have been considered, i.e. $\beta = 0{\cdot}8$ and $\beta = 0{\cdot}95$, respectively corresponding to a thick (10 cm) and to a thin (2·5 cm) covering sheet.

A typical flight path has been considered, represented in Fig. 1.3. The missile is launched at $t = 0$ with inclination of 60°, and constant acceleration of $7\,g$ for 26 sec; thus the altitude of 20,100 m is reached

($\simeq$66,000 ft) and the Mach number 6. Then the flight goes on at constant altitude and speed for 54 sec, where, with inclination 76°, descent is initiated at constant deceleration (7·57 g) in such a way as to reach sea level in 20 sec at $M = 1$.

Initial uniform temperature is 290°K ($\simeq$66°F).

Analysis and results. In this numerical example radiation was neglected; i.e. equation (1.I) was used with $\sigma = 0$.

The calculation of the coefficient of kinetic heating was performed by using the formula referring to the stagnation point value[3] for a sphere, in the laminar case, and for isothermal surfaces. The recovery temperature to be employed in Crocco's formula was computed by using, as recovery factor, the square root of the number of Prandtl, taken equal to 0·72.

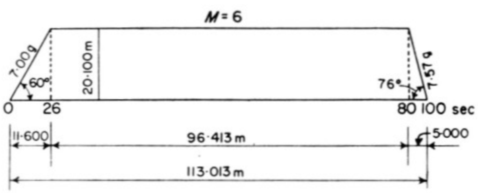


FIG. 2. Flight path.

Thus the boundary condition (1.I) is now written:

$$Q_W = h(T_W)[T_{aW} - T_W] \tag{1.3}$$

Of course, the case of radial flow only needed to be considered. For such case equation (13.1), written for Q = const., becomes:

$$T_W(t) = Q_W \left[\frac{3t}{1-\beta^3} + \frac{1}{1-\beta} \sum_{1}^{\infty}{}_n \frac{2\dfrac{p_n^2}{1+p_n^2}[1-\exp(-p_n^2 t)]}{1 - \dfrac{\cos p_n(1-\beta+2\phi_n)\cos p_n(1-\beta)}{1+\beta p_n^2}} \right] \tag{2.3}$$

Equations (1.3) (2.3) were employed in the step-by-step integration described in Part I, Art. 3.

Results are readily found, and reported in Fig. 3 that shows the timewise variation of skin temperature and internal temperature for both cases being considered. Results themselves are self-explanatory. Thermal stresses were also calculated (Figs. 4a, 4b) with the aid of formulas (253), ref. 12. Timewise variations of σ_r (radial stress), σ_θ (circumferential stress), show, first of all, a rapid increasing and a subsequent decreasing. This is obviously in connexion with the magnitude of time-gradients of temperature in the various regions of the head. It is, however, to be noticed that the maximum stresses are compressive stresses, which occur in the heated external region.

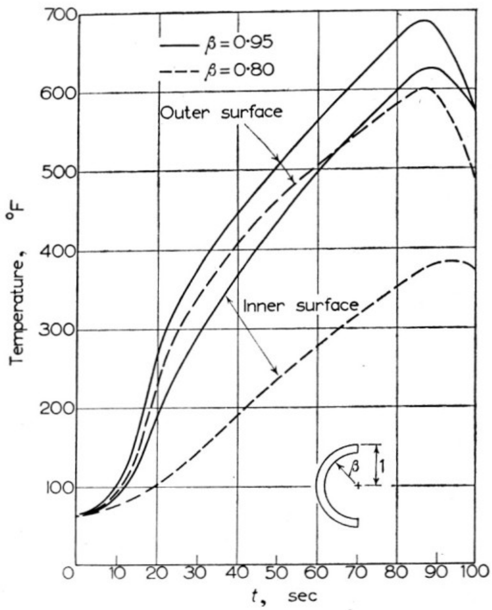


FIG. 3. Timewise variation of temperature in a hemisphere.

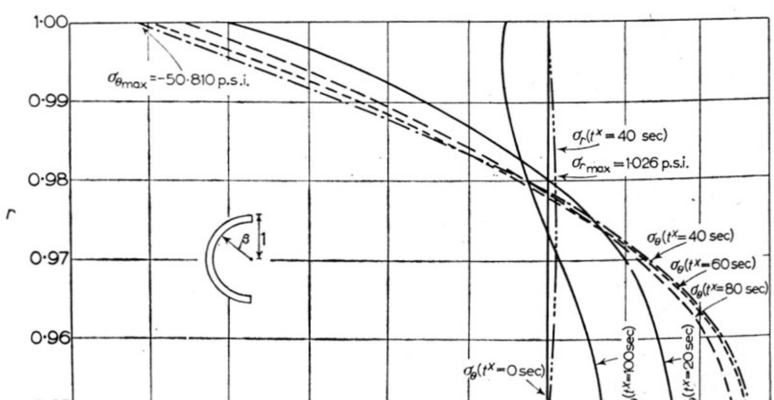


FIG. 4a. Thermal stresses along sheet thickness for hemisphere ($\beta = 0{\cdot}95$).

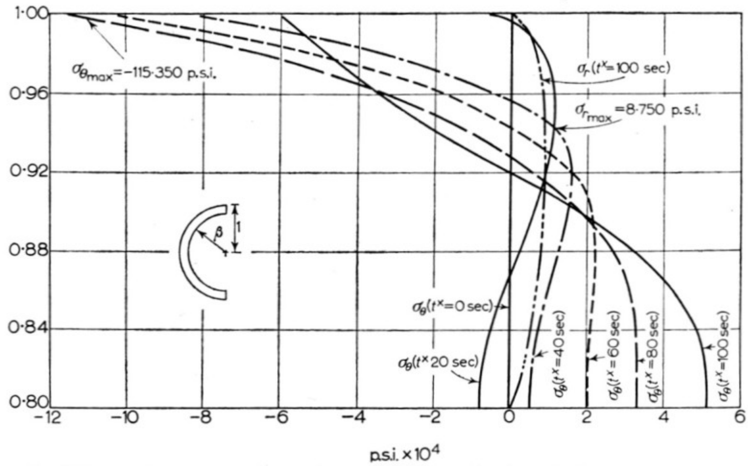


FIG. 4b. Thermal stresses along sheet thickness for hemisphere ($\beta = 0{\cdot}8$).

(b) *Transient Temperature Distribution in a Stiffener Section*

Geometrical feature and heat flows. The theory developed in Part 2, Art. 4, was applied to the section of Fig. 5, i.e. a symmetric L-section with $\lambda = \lambda' = \frac{5}{6}$; $b = 1$. This was done for the sake of comparison with the results of Ref. 10.

The heat flow is applied on the upper leg, is constant, and is taken as reference heat flow.

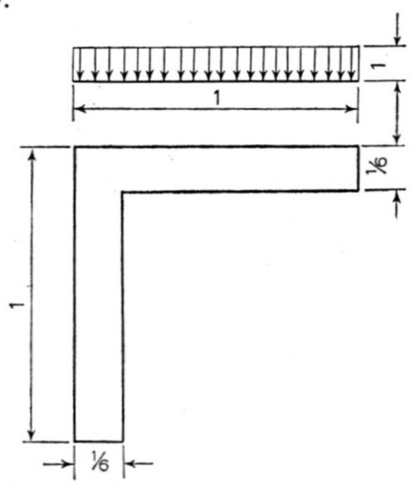


FIG. 5. Dimensions of stiffener section.

Eigenvalues. First of all, symmetric eigenfunctions were considered.

The functions $\Omega'(p)$ and $\Omega''(p)$ are decreasing with increasing p_r; thus the values of p are readily found by cutting the two quasi-straight lines:

$$\left.\begin{aligned} Y_1(p) &= p - \Omega'(p) + \Omega''(p) \\ Y_2(p) &= p - \Omega'(p) - \Omega''(p) \end{aligned}\right\} \qquad (3.3)$$

with horizontal straight lines $2j\pi$ and $(2j+1)\pi$ respectively (Fig. 6) (see also App. 2).

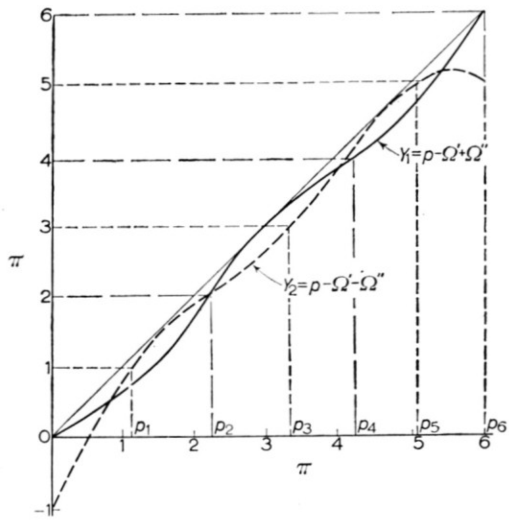


Fig. 6. Symmetric eigenvalues—Method of linearization.

Analogously antisymmetric eigenvalues were found.

The values of the roots are collected in Table 1.

It is seen that the number of roots smaller than 6π is sufficient for practical purposes.

Table 1

Eigenvalues

	Symmetric	Anti-Symmetric
n	$\frac{P_n}{\pi}$	$\frac{P_n}{\pi}$
0	0	0
1	1·088	0·567
2	2·165	1·704
3	3·200	2·829
4	4·180	3·933
5	5·086	5·023
6	6·000	6·000

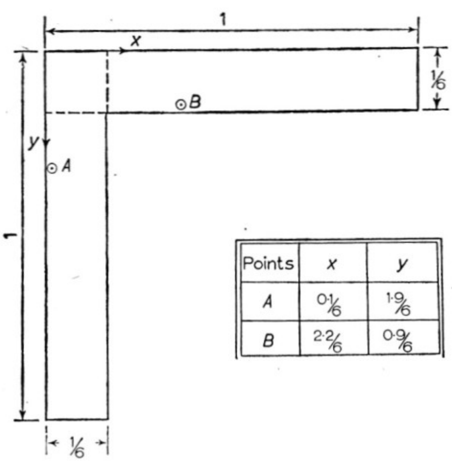
Final Results and Discussion

Figures 8 and 9 show time variation of temperature for points described in Fig. 7. Full lines show results obtained through the method here presented, dotted lines theoretical results obtained in Ref. 10. Points with star are results of tests conducted by electric analogue and also reported in Ref. 10.*

Results indicate a satisfactory agreement with experimental results. This can be better seen referring to the curve indicating the variation of T at $t = 4$ and $x = 0{\cdot}9$ for $0 \leqslant y \leqslant 1$. The agreement with experimental results (to be considered very accurate) is excellent in all the range (Fig. 9).

It is to be noticed that the entire calculation can be performed by one desk computer, by employing conventional electric computing machines, in about one week.

For the solution of Ref. 10, electronic machines were employed.



Points	x	y
A	0·1/6	1·9/6
B	2·2/6	0·9/6

FIG. 7. Points considered in comparisons.

* In Ref. 10, the length of the leg is taken 6 instead of unity (as done here). For the sake of comparison, corresponding changes in the value of the scale of non-dimensional times were made to the results here presented.

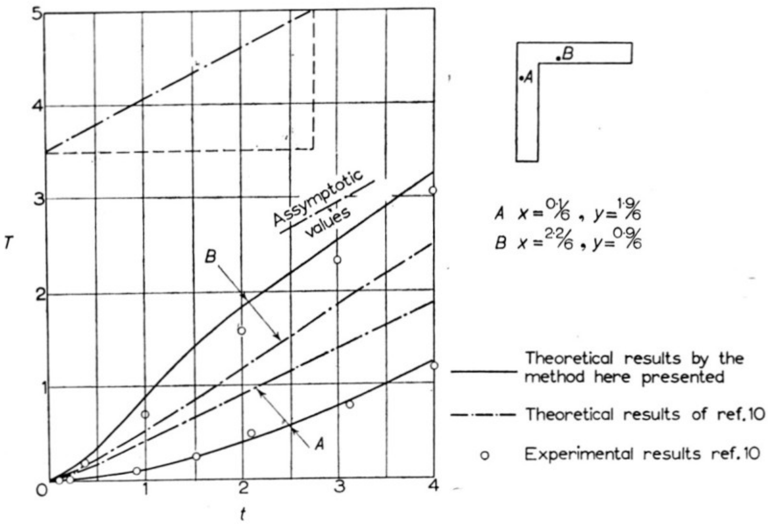


FIG. 8. Comparison with results of Ref. 10.
(Timewise variation of temperature.)

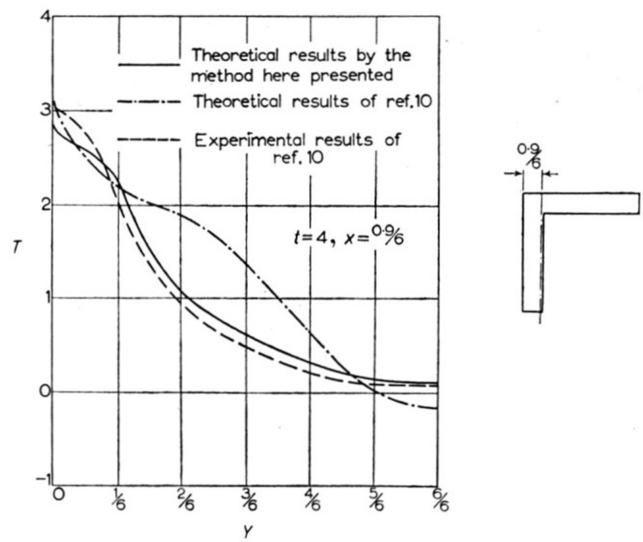


FIG. 9. Comparison with results of Ref. 10. (y-wise variation of temperature for $t = 4$, $x = 0{\cdot}9/6$.)

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